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Persistent Random Walks in a One-Dimensional Random Environment

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Central limit theorems are obtained for persistent random walks in a one-dimensional random environment. They also imply the central limit theorem for the motion of a test particle in an infinite equilibrium system of point particles where the free motion of particles is combined with a random collision mechanism and the velocities can take on three possible values.

KEY WORDS: Diffusion limit; persistent random walks; random environments.

1. INTRODUCTION

Recently it has become an intriguing problem to understand when random walks (r.w.) in a random environment (r.e.) converge to the Wiener process in the diffusion limit. Sinai's astonishing one-dimensional model[(11)] shows that, even in the presence of spatial symmetry in the definition of the environment, the behavior of a random walk in a random environment can be radically different from that of a classical random walk (in this model the displacement of the moving particle after n steps is of order $\log^2 n$ rather than $n^{1/2}$). It is, in general, expected that some form of symmetry of the transition operator, weaker than its self-adjointness will involve the central limit theorem (cf. Refs. 1, 7, 5 as partial confirmations) in arbitrary spatial dimension.

Our aim here is to investigate the asymptotic behavior of one-dimensional persistent random walks in a random environment.

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Suppose that at each site of the lattice $\mathbb{Z}$ a random scatterer is placed and it is characterized by left- and right-transpassing probabilities λ_j and μ_j $(j \in \mathbb{Z})$. These probabilities are chosen at random. For simplicity, we assume they are independent, identically distributed (i.i.d.) random variables. The collection of these random probabilities is the "environment."

Given the environment, a persistent random walk is a Markov chain X_n of order 2 with transition probabilities

$$\begin{aligned}
P(X_{n+1} = j+1 \mid X_{n-1} = j-1, X_n = j) &= \lambda_j \\
P(X_{n+1} = j-1 \mid X_{n-1} = j-1, X_n = j) &= 1 - \lambda_j \\
P(X_{n+1} = j-1 \mid X_{n-1} = j+1, X_n = j) &= \mu_j \\
P(X_{n+1} = j+1 \mid X_{n-1} = j+1, X_n = j) &= 1 - \mu_j
\end{aligned}$$

We show that if

$$\text{(S)} \quad \lambda_j = \mu_j \qquad \text{with probability 1} \qquad (j \in \mathbb{Z})$$

or

$$\text{(PD)} \quad \frac{\mu_j}{\lambda_j} \leqslant c < 1 \qquad \text{with probability 1} \qquad (j \in \mathbb{Z})$$

then the diffusion limit of the random walk in a random environment is a Wiener process. Though the transition operator of our random walk is not self-adjoint, the symmetry condition (S) turns out to be sufficient to obtain the result, while in the simpler case (PD) with positive drift we have an exponential mixing.

In Section 2 we give a more general formulation of the problem adding waiting times to the model and weakening the independence assumption on the environment, and we also state our theorems. Their proof is given in Section 4.

The motivation and the applicability of our results is discussed in Section 3. In fact, a one-dimensional infinite system of point particles moving uniformly with a random collision mechanism was investigated in equilibrium by Kipnis *et al.*[6] when the velocities had two possible values. By reducing the problem to a random central limit theorem they could show that the trajectory of a test particle is Wiener in the diffusion limit. In the same model, when the velocities have three possible values, we are able to represent the motion of the test particle as a random walk in a random environment, and thus it is shown to be asymptotically Wiener as a consequence of the results of Section 2. We expect our approach will work in a wider context, too.

2. EXACT FORMULATION AND MAIN RESULTS

Let $(\Omega, \mathscr{F}, \nu, T)$ be an ergodic dynamical system with an invertible measure preserving T. Suppose μ and λ are measurable functions on Ω taking values from the interval $[0, 1]$, and $\rho(ds \mid \pm 1, \cdot)$ are two transition probabilities from Ω to $\mathbb{R}_+$. The interpretation of these objects is the following. $\omega \in \Omega$ is identified with a realization of the environment; $T^k\omega$ is the kth translate of the same realization ($k \in \mathbb{Z}$). $\lambda(\omega)$ and $\mu(\omega)$ are the left- and right-transpassing probabilities of the scatterer placed at the origin in the realization ω. $\rho(ds \mid \pm 1, \omega)$ are the left and right waiting time distributions of the same scatterer.

We say that the environment is finitely dependent (F.D.) if $(\Omega, \mathscr{F})$ is a product space (i.e., $\Omega = \Omega_0^{\mathbb{Z}}$, where Ω_0 represents the random characteristics of the environment at one site), T is the left shift on it, ν is a measure with finite range of dependence (i.e., d dependent for some $d \geqslant 1$ in the language of probability theory), and the functions defined above depend only on the zeroth coordinate.

We shall consider two significant cases of the model: the *symmetric* one:

$$\text{(S)} \quad \lambda(\omega) = \mu(\omega) \qquad \nu\text{-a.s.} \tag{2.1}$$

and the case with *positive drift* (this naming will shortly become justified):

$$\text{(P.D.)} \quad \frac{\mu}{\lambda} \leqslant c < 1 \qquad \nu\text{-a.s.} \tag{2.2}$$

The process is defined in the following manner. $(v_n, \varepsilon_n, \tau_n)$ is a Markov chain on $\{+1, -1\} \times \Omega \times \mathbb{R}_+$, v_n being the nth jump of the random walker, ε_n the environment seen by the walker after this jump, and τ_n the nth waiting time.

The transition operator of this Markov chain is the following:

$$[\mathrm{Pf}](+1, \omega, t) = \lambda(\omega) \int \rho(ds \mid +1, T\omega) f(+1, T\omega, s)$$

$$+ [1 - \lambda(\omega)] \int \rho(ds \mid -1, T^{-1}\omega) f(-1, T^{-1}\omega, s)$$

$$[\mathrm{Pf}](-1, \omega, T) = [1 - \mu(\omega)] \int \rho(ds \mid +1, T\omega) f(+1, T\omega, s)$$

$$+ \mu(\omega) \int \rho(ds \mid -1, T^{-1}\omega) f(-1, T^{-1}\omega, s) \tag{2.3}$$

(v_n, ε_n) is also a Markov chain with transition operator

$$\begin{aligned}[P_0 f](+1, \omega) &= \lambda(\omega) f(+1, T\omega) + [1-\lambda(\omega)] f(-1, T^{-1}\omega) \\ [P_0 f](-1, \omega) &= [1-\mu(\omega)] f(+1, T\omega) + \lambda(\omega) f(-1, T^{-1}\omega)\end{aligned} \tag{2.4}$$

We shall denote by $m(t)$ the number of jumps effected until t:

$$m(t) = \min \left\{ n \;\middle|\; \sum_{k=0}^{n} \tau_k > t \right\} \tag{2.5}$$

and by $y(t)$ the position of the random walker at t:

$$y(t) = \sum_{k=1}^{m(t)} v_k \tag{2.6}$$

We shall prove invariance principle for this random function.

Theorem 1. In the symmetric case if

$$\lambda \leqslant a < 1 \qquad \nu\text{-a.s.} \tag{2.7}$$

and

$$\int \nu(d\omega)\, \lambda^{-1}(\omega) < \infty \tag{2.8}$$

then for almost all realizations of the environment

$$\zeta_A(t) = \frac{Y(At)}{\sqrt{A}} \Rightarrow W_\sigma \tag{2.9}$$

where W_σ is a Wiener process with zero mean and

$$\begin{aligned}\sigma^2 &= \left[\sum_{v=\pm 1} \frac{1}{2} \int \nu(d\omega) \int_0^\infty s p(ds/v, \omega) \right]^{-1} \left[\int \nu(d\omega) \frac{1-\lambda}{\lambda}(\omega) \right]^{-1} \\ &\overset{\text{def}}{=} \frac{1}{\langle \tau \rangle} \sigma_0^2\end{aligned} \tag{2.10}$$

The convergence $\Rightarrow$ of stochastic processes means the weak convergence in $D[0, \infty)$ supplied by the Skorohod topology (cf. Ref. 8).

Theorem 2. In the case with positive drift, if the environment is finitely dependent and if the waiting times have finite second moments (uniformly in ω), then with suitably chosen u and σ

$$\xi_A(t) = \frac{Y(At) - uAt}{\sqrt{A}} \Rightarrow W_\sigma \tag{2.11}$$

in probability with respect to the environment.

Remark. Some relatively simple calculations give

$$u = \frac{\langle v \rangle}{\langle \tau \rangle} \tag{2.12}$$

where

$$\langle v \rangle = \int v(d\omega)[\pi(+1, \omega) - \pi(-1, \omega)] \tag{2.13}$$

$$\langle \tau \rangle = \sum_{v = \pm 1} \int v(d\omega)\, \pi(v, \omega) \int s p(ds \mid v, \omega) \tag{2.14}$$

π is the density of the stationary distribution of the Markov chain (v_n, ε_n)

$$\pi(+1, \omega) = \frac{1 + \gamma(\omega)}{1 + 2\int v(d\omega)\, \gamma(\omega)}, \qquad \pi(-1, \omega) = \frac{\gamma(T\omega)}{1 + 2\int \beta(d\omega)\, \gamma(\omega)} \tag{2.15}$$

with

$$\gamma(\omega) = \frac{1-\lambda}{\lambda}(\omega) + \frac{\mu}{\lambda}(\omega)\left\{\frac{1-\lambda}{\lambda}(T\omega) + \frac{\mu}{\lambda}(T\omega)\left[\frac{1-\lambda}{\lambda}(T^2\omega) + \frac{\mu}{\lambda}(T^2\omega)\cdots\right.\right. \tag{2.16}$$

The calculation of σ seems more complicated.

3. APPLICATIONS

In this section we shall give some more physical models which the theorems presented above can be applied to. In each example we couple a random walk to the one-dimensional motion of a test particle.

3.1. The Random Collision Model

This model is a modification of Harris' one-dimensional hard point gas.[4,13] An infinite system of identical point particles is distributed on the line according to Poisson distribution with density 1. The velocities of the particles are i.i.d. random variables with mean zero. The particles move freely except collisions: when two particles meet each other they collide elastically with probability $1-\pi$ (i.e., with probability π they pass through each other conserving their initial velocities; with probability $1-\pi$ they exchange their velocities.) We are interested in describing the trajectory of a tagged particle in the system. The sure collision model ($\pi = 0$) was exhaustively treated by Spitzer[13] in equilibrium and by Major and Szász[10] in the nonequilibrium case. Major and Szász found that in the general nonequilibrium case the trajectory of a tagged particle normed in the standard way tends in distribution to a Gaussian process which is, in general, not Wiener.

We can apply our theorems to the case where the particles are in equilibrium distribution (Poisson) and the velocity distribution is concentrated on three points:

$$\alpha = -, 0, +, \qquad \mathscr{P}(v_\alpha) = p_\alpha, \qquad \sum_\alpha p_\alpha = 1, \qquad \sum_\alpha v_\alpha p_\alpha = 0 \tag{3.1}$$

Denoting the trajectory of a tagged particle by $y(t)$ we can prove that with a suitably chosen constant σ

$$\frac{y(At) - y(0)}{\sigma \sqrt{A}} \Rightarrow W(t) \tag{3.2}$$

where $W(t)$ is a standard Wiener process. If $v_0 = 0$ (symmetric case) we can also calculate

$$\sigma^2 = \frac{1+\pi}{(1-\pi)[1+\pi(1-p_0)]} E\,|v| \tag{3.3}$$

We give the details of the reduction to the coupled random walk only for the symmetric case, the other one being essentially the same.

The positions of the zero-velocity particles are fixed once and for ever: $\{S_i\}_{i \in \mathbb{Z}}$ in a natural order. The random variables $\xi_i = S_{i+1} - S_i$ are i.i.d. exponentially distributed with parameter p_0. We shall consider the random walk on $\mathbb{Z}$ coupled with the motion of the tagged particle in the following way: the walking particle jumps to the lattice site i when the tagged particle enters the open interval (S_i, S_{i+1}) and stays there until the tagged particle leaves the closed interval $[S_i, S_{i+1}]$.

The "environment space" will be

$$\Omega = \mathbb{R}_+^{\mathbb{Z}} = \{(\dots, l_{-1}, l_0, l_1, \dots) \mid l_i \in \mathbb{R}_+\} \tag{3.4}$$

with the product σ-algebra and

$$\nu = \bigotimes_{i=-\infty}^{\infty} m_i \tag{3.5}$$

with each m_i exponentially distributed with parameter p_0. T is the left shift on Ω. The coupled random walk is trivially a persistent r.w. in r.e. with waiting times.

One can calculate

$$\frac{1-\lambda}{\lambda}(..., l_0,...) = \frac{1-\pi}{2}[1 + (1+\pi)(1-p_0)l_0] \tag{3.6}$$

$$\langle \tau \rangle = \frac{2}{(1-\pi)\, p_0 E\, |v|} \tag{3.7}$$

The result is modified by the fact that the interval lengths are random:

$$\frac{Y(At)}{\sqrt{A}} = \frac{Y(At)}{X(At)+\varepsilon} \frac{X(At)+\varepsilon}{\sqrt{A}} \tag{3.8}$$

In (3.8) $X(t)$ is the lattice position of the random walker (we have to add ε to avoid zero in the denominator). Due to the fact that $|X(At)|$ tends to infinity in probability, the first factor tends in probability to $\langle \xi \rangle$, the average of the interval length. Consequently, by Theorem 1 we have the desired result.

3.2. The Stochastic Lorentz Gas

An almost trivial application of Theorem 1 is the one-dimensional stochastic Lorentz gas: random scatterers are placed on the line according to some translation invariant, ergodic distribution. Each scatterer has a transpassing probability λ. A particle with unit velocity is moving on the line. When it arrives to a scatterer it decides to continue its way or to turn back according to the respective transpassing probability. After trivial calculations one finds the desired result with

$$\sigma^2 = \langle \xi \rangle \left\langle \frac{1-\lambda}{\lambda} \right\rangle^{-1} \tag{3.9}$$

where $\langle \xi \rangle$ is the man value of the spacing between two neighboring scatterers.

4. PROOFS

Proof of Theorem 1. In the symmetric case the Markov chain $(v_n, \varepsilon_n, \tau_n)$ has the following stationary distribution:

$$\rho(ds \mid v, \omega)\, \pi(v, \omega)\, \nu(d\omega), \;\; \pi(v, \omega) = 1/2 \tag{4.1}$$

The stationary Markov chain is also ergodic because—due to the ergodicity of the environment—1 is a nondegenerate eigenvalue of P_0. One can easily check this in the Hilbert space $L_2(\{-1, 1\} \times \Omega, \pi dv)$. (See Appendix.) Consequently the strong law of large numbers is true for any function of the Markov chain integrable with respect to $\rho\pi\, dv$. In particular

$$\frac{t}{m(t)} \xrightarrow{\text{a.s.}} \langle\tau\rangle = \frac{1}{2} \sum_{v=\pm 1} \int \nu(d\omega) \int_0^\infty s\rho(ds \mid v, \omega) \tag{4.2}$$

Now we turn to the invariance principle:

$$\frac{Y(At)}{\sqrt{A}} = \frac{\sum_{k=1}^{m(At)} v_k}{\sqrt{A}} \tag{4.3}$$

Due to (4.2) and Theorem 17.1 of Ref. 2, on sums with a random number of summands it is sufficient ro prove the invariance principle for $(\sum_{k=1}^{[At]} v_k)/\sqrt{A}$ (that is, for the case without random waiting times).

A second remark is that it suffices to prove this for the case when $\lambda < 1/2$ a.s. If we have the weaker condition (2.7) we form blocks of N consecutive scatterers and identify each block with one site of a "super lattice" and observe the random walk on this. The essential observation is that this random walk will also be a symmetric persistent one with waiting times where

$$\tilde{\lambda}_i = \left[\sum_{j \in \text{block}_i} \left(\frac{1}{\lambda_j} - 1 \right) + 1 \right]^{-1} \tag{4.4}$$

and

$$\langle\tau\rangle = N \tag{4.5}$$

By (4.4) we can choose such an N that $\tilde{\lambda} < 1/2$ a.s.

If $\lambda < 1/2$ we prove the theorem by choosing a convenient realization of the random walk. Given the environment ω, fix the following point system on the real line:

$$A_0 = -\frac{1-2\lambda(\omega)}{\lambda(\omega)}, \qquad B_0 = -A_0$$

$$A_i = B_{i-1} + 1, \qquad B_i = A_i + \frac{1-2\lambda(T^i\omega)}{\lambda(T^i\omega)} \qquad (i \geqslant 1)$$

$$B_i = A_{i+1} - 1, \qquad A_i = B_i - \frac{1-2\lambda(T^i\omega)}{\lambda(T^i\omega)} \qquad (i \leqslant -1) \tag{4.6}$$

Consider a Wiener process starting from the origin and couple with its motion a random walk on the lattice $\mathbb{Z}$ in the following way: the random walker jumps to the site i when the Brownian particle enters the closed interval $[A_i, B_i]$. This r.w. will be a persistent one with waiting times with the same realization of the environment. Let us denote by T_N the sum of the first N waiting times and by $W(t)$ the trajectory of the Brownian particle. We have

$$\frac{\sum^{[At]} v_k}{\sqrt{A}} = \frac{\sum^{[t]} v_k}{W_{T_{[At]}}} \frac{W_{T_{[At]}}}{\sqrt{A}} \tag{4.7}$$

But $W_{T_{[At]}}$ is equal to $A_{\sum^{[At]} v_k}$ or $B_{\sum^{[At]} v_k}$ and $|\sum^{[At]} v_k| \to \infty$ in probability. So for the first factor we have the weak law of large numbers:

$$\frac{\sum^{[At]} v_k}{W_{T_{[At]}}} \xrightarrow{P} \left[\int v(d\omega) \frac{1-\lambda}{\lambda}(\omega)\right]^{-1} \tag{4.8}$$

we also have

$$\frac{T_{[At]}}{At} \xrightarrow{\text{a.s.}} \int v(d\omega) \frac{1-\lambda}{\lambda}(\omega) \tag{4.9}$$

consequently by Theorem 4.A of Ref. 3 we have the desired result. ∎

Proof of Theorem 2. For the sake of completeness we mention that, in the P.D. case, the Markov chain has also an absolutely continuous stationary distribution of the form (4.1) with π given in (2.15), (2.16), but we do not use this fact in the proof.

We have to introduce the following auxiliary random variables:

$$Z(t) = \max_{s \leqslant t} Y(s) \tag{4.10}$$

$$T_i = \min\{t \mid Y(t) = i\}, \qquad i \geqslant 0 \tag{4.11}$$

$$\theta_i = T_i - T_{i-1}, \qquad i \geqslant 1 \tag{4.12}$$

$$K_i = \max\{i - Y(s) \mid T_i \leqslant s < T_{i+1}\} \tag{4.13}$$

$$\delta_1(t) = t - T_{Z(t)} \tag{4.14}$$

$$\delta_2(t) = Y(t) - Z(t) \tag{4.15}$$

The physical meaning of these variables is transparent.

By stationarity of the environment $\{\theta_i\}_{i \geqslant 1}$ and $\{K_i\}_{i \geqslant 1}$ are stationary sequences of random variables. Due to the fact that the waiting times have finite second moments, θ_i have too. Due to (2.2) K_i have moments of all orders. The following inequalities are evident:

$$|\delta_1(t)| \leqslant \theta_{Z(t)}, \qquad |\delta_2(t)| \leqslant K_{Z(t)} \tag{4.16}$$

The crucial observation is that the random variables $\{\theta_i\}$ are exponentially α-mixing. Let $\mathscr{F}_k$ and $\mathscr{F}^k$ $(k \geqslant 1)$ be the σ-algebras generated by $\{\theta_i\}_{i=1}^k$ and $\{\theta_i\}_{i=k}^\infty$, respectively, and $M_{k,m}$ the event that the random walker comes back from site $k+m$ to site $k+d, d$ being the range of dependence of the environment $(m > d)$. If $A \in \mathscr{F}_k$, $B \in \mathscr{F}^{k+m}$, simple calcultions show that

$$\begin{aligned} |\mathscr{P}(B \mid A) - \mathscr{P}(B)| &= |\mathscr{P}(B \mid A \cap M_{k,m}) - \mathscr{P}(B \mid M_{k,m}) \mid \mathscr{P}(M_{k,m}) \\ &\leqslant 2\mathscr{P}(M_{k,m}) \end{aligned} \tag{4.17}$$

The right-hand side of (4.17) is explicitly calculable, and due to (2.2) one finds that it is exponentially small. Consequently

$$\frac{Z(t)}{t} \xrightarrow{\text{a.s.}} \frac{1}{E\theta} = u \tag{4.18}$$

Turning to the invariance principle we have

$$\frac{Y(At) - uAt}{\sqrt{A}} = u \frac{\sum_{i=1}^{Z(At)} (1/u - \theta_i)}{\sqrt{A}} + \frac{\delta_2(At) - u\delta_1(At)}{\sqrt{A}} \tag{4.19}$$

Due to (4.16) the last term is dominated by

$$\frac{k_{Z(At)} + u\theta_{Z(At)}}{\sqrt{A}} \tag{4.20}$$

This expression tends in probability to zero uniformly in $t \in (0, 1]$ because

$$\begin{aligned}
&\mathscr{P}\left(\sup_{k<Z(A)} \theta_k > \eta\sqrt{A}\right) \\
&\quad = \mathscr{P}\left(\sup_{k<Z(A)} \theta_k > \eta\sqrt{A} \;\middle|\; \frac{Z(A)}{A} < u+\varepsilon\right) \mathscr{P}\left(\frac{Z(A)}{A} < u+\varepsilon\right) \\
&\qquad + \mathscr{P}\left(\sup_{k<Z(A)} \theta_k > \eta\sqrt{A} \;\middle|\; \frac{Z(A)}{A} \geqslant u+\varepsilon\right) \mathscr{P}\left(\frac{Z(A)}{A} \geqslant u+\varepsilon\right) \\
&\quad \leqslant \mathscr{P}\left(\sup_{k<A(n+\varepsilon)} \theta_k > \eta\sqrt{A}\right) + \mathscr{P}\left(\frac{Z(A)}{A} \geqslant u+\varepsilon\right) \\
&\quad \leqslant (u+\varepsilon) E(\theta^2 \chi(\theta > \eta\sqrt{A})) + \mathscr{P}\left(\frac{Z(A)}{A} \geqslant u+\varepsilon\right) \to 0
\end{aligned} \tag{4.21}$$

for any η. The same argument applies to $k_{Z(At)}$. Applying Theorem 20.1 of Ref. 2 to the first term in (4.19) and then once again Theorem 17.1 of the same reference we obtain the desired result.

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APPENDIX

We shall prove that, in the symmetric case, 1 is a nondegenerate eigenvalue of P_0 considered as an operator on $L_2(\{+1, -1\} \times \Omega, \pi dv)$. For this purpose we write

$$P_0 = AU \tag{A.1}$$

with

$$\begin{aligned}
[Af](v, \omega) &= \lambda(\omega) f(v, \omega) + [1 - \lambda(\omega)] f(-v, \omega) \\
[Uf](v, \omega) &= f(v, T^v \omega)
\end{aligned} \tag{A.2}$$

U is unitary and, by using the fact that λ is not concentrated to 0 and 1,

$$\|Af\| < \|f\| \tag{A.3}$$

for any nonconstant f. Consequently

$$\|P_0 f\| < \|f\| \tag{A.4}$$

for any nonconstant f.

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